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Temperature and the Critical Dimension of Strings

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Thermofield dynamics is used to generalize the BRST invariance of strings to nonzero temperature. The requirement that the BRST generator is nilpotent implies that $d = 26$ even at nonzero temperature.

1. INTRODUCTION

String theory is the best candidate for unifying the known forces. Since its role becomes significant at high temperature, the study of finite-temperature string theory is of great importance. One property of strings is the existence of a critical dimension of space-time at which the Lorentz algebra closes in the light cone gauge (Schwarz, 1982), the conformal anomaly cancels (Friedan *et al.* 1986), and the BRST charge is nilpotent (Kato and Ogawa, 1983).

The question in which we are interested is: Can temperature affect the critical dimension of strings? This question is important for the compactification of strings, as we discuss in Section 5. To answer this question one has to generalize the BRST symmetry or equivalently the conformal symmetry to finite temperature. BRST invariance at finite temperature for gauge theory has been studied (Ojima, 1981; Matsumoto *et al.* 1983), using thermofield dynamics (TFD) (Takahashi and Umezawa, 1975). Recently TFD has been applied to strings (Ahmed *et al.*, 1987) and introduced in superstrings (Ahmed, 1987).

Section 2 gives a brief review of BRST symmetry of strings. Section 3 presents the basic formalism of TFD and its application in gauge theory. In Section 4 TFD is applied in strings to derive the critical dimension of strings at finite temperature. Section 5 presents our conclusions.

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2. STRING BRST SYMMETRY AT $T=0$

The closed bosonic string is described by the action

$$S=\frac{-1}{2\pi}\int d\sigma\, d\tau\, (-g)^{1/2} g^{\alpha\beta}\, \partial_\alpha X^\mu\, \partial_\beta X_\mu \tag{2.1}$$

After fixing the 2D reparametrization invariance one gets

$$\begin{aligned} X^\mu &= (X_L^\mu(\tau-\sigma)+X_R^\mu(\tau+\sigma)) \\ X_R^\mu &= x^\mu+\tfrac{1}{2}P^\mu(\tau+\sigma)+\frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^\mu\, e^{-in(\tau+\sigma)} \\ [\alpha_n^\mu, \alpha_m^\nu] &= n\,\delta_{n+m,0}\eta^{\mu\nu} \end{aligned} \tag{2.2}$$

We also obtain the set of constraints

$$L_n=\tfrac{1}{2}\sum_{m=-\infty}^{\infty}\alpha_{n-m}^\mu \alpha_m^\mu \tag{2.3}$$

Following the usual Fadeev-Popov procedure, we introduce the ghosts C_n, $\bar{C}_m$, which satisfy

$$\begin{aligned} \{C_n, C_m\} &= \{\bar{C}_n, \bar{C}_m\}=0 \\ \{C_n, \bar{C}_m\} &= \delta_{n+m,0} \\ C_n^\dagger = C_{-n}, &\qquad \bar{C}_n^\dagger=\bar{C}_{-n} \end{aligned} \tag{2.4}$$

The BRST charge Q_B is given by

$$Q_B=\sum_{n=-\infty}^{\infty} C_n L_{-n}-\tfrac{1}{2}\sum_{m,n=-\infty}^{\infty}(m-n):C_{-n}C_{-m}\bar{C}_{n+m}:-\alpha C_0 \tag{2.5}$$

The ghost charge is

$$iQ_c=\tfrac{1}{2}(C_0\bar{C}_0-\bar{C}_0C_0)+\sum_{n\neq 0}:C_{-n}\bar{C}_n: \tag{2.6}$$

They satisfy:

$$[iQ_c, Q_B]=Q_B \tag{2.7}$$

and

$$Q_B^2=\tfrac{1}{2}\{Q_B, Q_B\}=0 \tag{2.8}$$

which is satisfied only if

$$d=26, \qquad \alpha=1 \tag{2.9}$$

Conditions (2.7) and (2.8) are crucial for the physical state subsidiary condition:

$$Q_B|\text{phys}\rangle=0 \tag{2.10}$$

3. THERMOFIELD DYNAMICS

The basic idea of TFD is to express the statistical average of any operator A as a vacuum expectation value with the temperature-dependent vacuum $|0(\beta)\rangle$,

$$\operatorname{tr}(e^{-\beta H}A)/\operatorname{tr}(e^{-\beta H}) = \langle 0(\beta)|A|0(\beta)\rangle \tag{3.1}$$

This requires a doubling of the field degrees of freedom. This is achieved by defining for every field operator $\mathscr{A}$ a tilde conjugate $\tilde{\mathscr{A}}$ such that

$$\begin{gathered}(\widetilde{\mathscr{A}_1\mathscr{A}_2}) = \tilde{\mathscr{A}}_1\tilde{\mathscr{A}}_2 \\ (\widetilde{\lambda_1\mathscr{A}_1+\lambda_2\mathscr{A}_2}) = \lambda_1^*\tilde{\mathscr{A}}_1+\lambda_2^*\tilde{\mathscr{A}}_2 \\ \tilde{\tilde{\mathscr{A}}} = \pm\mathscr{A}, \qquad +(-) \text{ for bosons (fermions)}\end{gathered} \tag{3.2}$$

For every field ϕ one now has a doublet ϕ^α, $\alpha = 1, 2$, defined by

$$\begin{bmatrix}\phi^1\\ \phi^2\end{bmatrix} = \begin{bmatrix}\phi\\ \tilde{\phi}^\dagger\end{bmatrix}, \qquad \begin{bmatrix}\psi^1\\ \psi^2\end{bmatrix} = \begin{bmatrix}\psi\\ C\tilde{\bar{\psi}}^t\end{bmatrix} \tag{3.3}$$

where ψ is a spinor field, t denotes the transpose, $\bar{\psi} = \psi^\dagger\gamma^0$, and C is the charge conjugation. The total Lagrangian is given by

$$\bar{\mathscr{L}} = \mathscr{L} - \tilde{\mathscr{L}} = \sum_{\alpha=1}^{2} \varepsilon_\alpha \mathscr{L}_\alpha \tag{3.4}$$

where

$$\varepsilon_\alpha = \begin{cases} 1 & \alpha = 1 \\ -1 & \alpha = 2\end{cases}$$

$$\mathscr{L}_\alpha = P_\alpha \mathscr{L}(\phi, \psi)$$

$$P_\alpha(A^\alpha B^\alpha \cdots C^\alpha) \equiv \begin{cases} A^1B^1\cdots C^1 & \alpha = 1 \\ C^2\cdots B^2A^2 & \alpha = 2\end{cases}$$

The thermal fields ϕ_β^α and ψ_β^α are introduced by the Bogoliubov transformation

$$\begin{gathered}\phi^\alpha_{(x)} = [U_B(-i\,\partial/\partial t)]^{\alpha\gamma}\phi_\beta^\gamma(x) \\ \psi^\alpha(x) = [U_F(-i\,\partial/\partial t)]^{\alpha\gamma}\psi_\beta^\gamma(x)\end{gathered} \tag{3.5}$$

where

$$\begin{gathered}U_B(w) = (e^{\beta\omega}-1)^{1/2}\begin{bmatrix} e^{\beta\omega/2} & 1 \\ 1 & e^{\beta\omega/2}\end{bmatrix} \\ U_F(\omega) = (e^{\beta\omega}+1)^{-1/2}\begin{bmatrix} e^{\beta\omega/2} & 1 \\ -1 & e^{\beta\omega/2}\end{bmatrix}\end{gathered} \tag{3.6}$$

The coefficients of the normal mode expansion of these thermal fields are the creation and annihilation operators of the thermal vacuum $|0(\beta)\rangle$.

We apply TFD to the gauge theory described by the action

$$S=\int dx\,[-\tfrac{1}{4}F^a_{\mu\nu}F^a_{\mu\nu}+\partial_\mu\bar{C}^a(D^\mu C)^a-A^a_\mu\partial^\mu B^a+\tfrac{1}{2}B^aB^a] \tag{3.7}$$

where C and $\bar{C}$ are Fadeev–Popov ghost fields, B is a Lagrange multiplier, and

$$F^a_{\mu\nu}=\partial_\mu A^a_\nu-\partial_\nu A^a_\mu-igf^{abc}A^b_\mu A^c_\nu$$

$$(D_\mu C)^a=\partial_\mu C^a-igf^{abc}A^b_\mu C^c$$

The thermal Lagrangian $\bar{\mathscr{L}}$ is given by

$$\bar{\mathscr{L}}=\mathscr{L}-\tilde{\mathscr{L}}$$

$$\tilde{\mathscr{L}}=-\tfrac{1}{4}\tilde{F}^a_{\mu\nu}\tilde{F}^a_{\mu\nu}+(\partial_\mu\tilde{\bar{C}})^a(D^\mu\tilde{C})^a-\tilde{A}^a_\mu\partial^\mu\tilde{B}^a+\tfrac{1}{2}\tilde{B}^a\tilde{B}^a \tag{3.8}$$

It has been shown that the ghost fields behave like bosons under Bogoliubov transformations.

The finite-temperature ghost and BRST charges are

$$\bar{Q}_c=i\int d^{D-1}_x\,[\bar{C}^a(D_0C)^a-(\partial_0\bar{C})C^a]-\text{tilde conjugate} \tag{3.9}$$

$$\bar{Q}_B=\int d^{D-1}_x\,[B^a(D_0C)^a-(\partial_0B^a)C^a+\tfrac{1}{2}igf^{abc}(\partial_0\bar{C})^aC^bC^c]-\text{tilde conjugate} \tag{3.10}$$

They satisfy

$$\bar{Q}^2_B=0 \tag{3.11}$$

$$[i\bar{Q}_c,\bar{Q}_B]=\bar{Q}_B \tag{3.12}$$

Equations (3.11) and (3.12) ensure that the quartet mechanism is valid at $T\neq 0$; consequently, the physical state subsidiary condition is

$$\bar{Q}_B|\text{Phys}\rangle=0 \tag{3.13}$$

4. STRINGS AT FINITE TEMPERATURE

Now we can formulate the BRST symmetry of string theory at finite temperature. The thermal action is given by

$$S=\frac{-1}{2\pi}\int d\sigma\,d\tau\,[(-g)^{1/2}g^{\alpha\beta}\,\partial_\alpha X^\mu\,\partial_\beta X_\mu-(-\tilde{g})^{1/2}\tilde{g}^{\alpha\beta}\,\partial_\alpha\tilde{X}^\mu\,\partial_\beta\tilde{X}_\mu] \tag{4.1}$$

where $\tilde{X}_\mu$ is the tilde conjugate of X_μ. Notice that the 2D tilde metric $\tilde{g}_{\alpha\beta}$ in general has to be different from $g_{\alpha\beta}$. This can be seen by realizing that in TFD each $\mathscr{L}$ and $\tilde{\mathscr{L}}$ has its own gauge invariance. Furthermore, one of the known choices of the metric $g_{\alpha\beta}$ is

$$g_{\alpha\beta} = \partial_\alpha X^\mu \partial_\beta X_\mu \neq \partial_\alpha \tilde{X}^\mu \partial_\beta \tilde{X}_\mu = \tilde{g}_{\alpha\beta} \tag{4.2}$$

If one uses the light cone gauge, then (4.1) reduces to the known thermal action in the light cone. In this paper we use the covariant formulation of strings.

Varying the action (4.1) with respect to $g_{\alpha\beta}$ and $\tilde{g}_{\alpha\beta}$ gives the constraints

$$\tilde{L}_m = 0, \qquad L_m = 0$$

$$\tilde{L}_m = \tfrac{1}{2} \sum_{n=-\infty}^{\infty} \tilde{\alpha}^\mu_{m-n} \tilde{\alpha}^\mu_n \tag{4.3}$$

where $\tilde{\alpha}^\mu_n$ are the normal mode coefficients of $\tilde{X}^\mu$.

Introducing Fadeev–Popov ghosts for both types of the constraints (4.3), we derive the thermal ghost number charge $i\bar{Q}_c$ and BRST charge $\bar{Q}_B$:

$$i\bar{Q}_c = iQ_c - i\tilde{Q}_c$$

$$i\tilde{Q}_c = \tfrac{1}{2}(\tilde{C}_0 \tilde{\bar{C}}_0 - \tilde{\bar{C}}_0 C_0) - \tfrac{1}{2} \sum_{n=1}^{\infty} (\tilde{C}_{-n} \tilde{\bar{C}}_n - \tilde{\bar{C}}_{-n} \tilde{C}_n)$$

$$\{\tilde{C}_n, \tilde{\bar{C}}_m\} = -\delta_{n+m,0} \tag{4.4}$$

$$\bar{Q}_B = Q_B - \tilde{Q}_B \tag{4.5}$$

$$\tilde{Q}_B = \sum \tilde{C}_n L_{-n} - \tfrac{1}{2} \sum_{n,m=-\infty}^{\infty} (m-n) : \tilde{C}_{-n} \tilde{C}_{-m} \tilde{\bar{C}}_{n+m} : - \tilde{\alpha} \tilde{C}_0$$

$$[i\tilde{Q}_c, \tilde{Q}_B] = -\tilde{Q}_B \tag{4.6}$$

$$\{Q_B, \tilde{Q}_B\} = 0 \tag{4.7}$$

It is easy to check that

$$[i\bar{Q}_c, \bar{Q}_B] = \bar{Q}_B \tag{4.8}$$

The requirement that the BRST charge is nilpotent, $\bar{Q}_B^2 = 0$, and the relation (4.7) imply

$$\bar{Q}_B^2 = 0 \Leftrightarrow Q_B^2 = 0, \quad \tilde{Q}_B^2 = 0 \tag{4.9}$$

which is satisfied only if

$$d = 26, \qquad \alpha = 1 \tag{4.10}$$

Therefore we conclude that temperature will not affect the space–time dimension of strings.

From the relations (4.8) and (4.9) we deduce that the physical state subsidary condition at finite temperature for strings is

$$\bar{Q}_B|\text{phys}\rangle = 0 \tag{4.11}$$

5. CONCLUSIONS

Thermofield dynamics has been used to investigate the BRST symmetry of strings at finite temperature. The nilpotency of the BRST charge implies that temperature will not affect the critical dimension of strings. We anticipate that this result will be true for superstrings as well. The results presented here should be of basic importance to the problem of compactification of strings and superstrings because some of the compactification manifolds, e.g., Calabi-Yau manifolds (Green *et al.*, 1987), are defined only in $d_{\text{critical}} - 4$ dimensions. We expect that temperature will not spoil the results obtained there.

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